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An Intriguing Legacy of Einstein, Fermi, Jordan, and Others: The Possible Invalidation of Quark Conjectures[1]

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The objective of this paper is to present an outline of a number of criticisms of the quark models of hadron structure which have been present in the community of basic research for some time. The hope is that quark supporters will consider these criticisms and present possible counterarguments for a scientifically effective resolution of the issues. In particular, it is submitted that the problem of whether quarks exist as physical particles necessarily calls for the prior theoretical and experimental resolution of the question of the validity or invalidity, for hadronic structure, of the relativity and quantum mechanical laws established for atomic structure. The current theoretical studies leading to the conclusion that they are invalid are considered, together with the experimental situation. We also recall the doubts by Einstein, Fermi, Jordan, and others on the final character of contemporary physical knowledge. Most of all, this paper is an appeal to young minds of all ages. The possible invalidity for the strong interactions of the physical laws of the electromagnetic interactions, rather than constituting a scientific drawback, represents instead an invaluable impetus toward the search for covering laws specifically conceived for hadronic structure and strong interactions in general, a program which has already been initiated by a number of researchers. In turn, this situation appears to have all the ingredients for a new scientific renaissance, perhaps comparable to that of the early part of this century.

1. THE QUARK MODELS

Truly outstanding achievements have occurred in the study of the strongly interacting particles (hadrons) during the last decades. Beginning with the pioneering proposal by Gell-Mann[(1)] and Zweigh[(2)] of using the special

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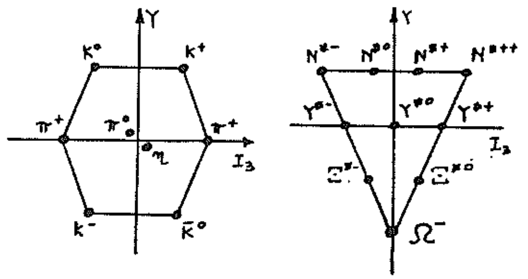

Fig. 1

unitary Lie group $SU(3)$ for the study of hadrons, the celebrated "eightfold way" provided the first comprehensive treatment of hadrons, with achievements such as the octet representation of the light mesons and the decuplet representation of baryons (Fig. 1). In particular, this approach to hadrons predicted the existence of the particle Ω^-, which was subsequently established by experiments.

The decomposition of the tensorial product of the fundamental three-dimensional representation of $SU(3)$ and its conjugate $\bar{3}$

$$3 \times \bar{3} = 8 + 1, \qquad 3 \times 3 \times 3 = 10 + 8 + 8 + 1 \tag{1}$$

suggested the hypothesis that mesons are a bound state of one quark and an antiquark, while baryons are a bound state of three quarks.

This original hypothesis, later called "the naive quark model," was subjected to a number of sequential, conceptual, and technical implementations. We mention only (i) the introduction of the so-called "sea of gluons" to attempt a field-theoretic description of quark models; (ii) the introduction of the notion of "color," in order to bypass an inconsistency created by the spin-statistics theorem for the Δ^{++} state (among other reasons); and (iii) the reformulation of the quark models in terms of so-called "bags" (e.g., the MIT and SLAC bag models).

More recently, a brilliant advance was achieved via the introduction of a new notion, called "charm," and new quarks. These advances led to the discovery of a new heavy hadron, the J/ψ particle, and, still more recently, to a new series of heavy hadrons sometimes referred to as "charmonium states."[3]

By incorporating a number of outstanding achievements (such as the non-Abelian gauge theory by Yang and Mills and the unified gauge theory of weak and electromagnetic interactions by Salam and Weinberg), a comprehensive theory of hadrons, called quantum chromodynamics (QCD), was proposed. According to this theory, hadron physics is described by the Lagrangian (density)[(5,6)4]

$$\mathscr{L}_{\text{tot}} = \mathscr{L}_{\text{free}} + \mathscr{L}_{\text{int}} = -\tfrac{1}{4}G_a^{\mu\nu}G_{a\,\nu} + i(\bar{\psi}_q D\psi_q - m\bar{\psi}_q\psi_q) \tag{2}$$

whose fields satisfy the Lorentz covariance law

$$\psi'(x') = S(\Lambda)\psi(x), \qquad \Lambda \in SL(2.C) \tag{3}$$

as well as the unitary gauge symmetry. The results achieved by QCD are equally impressive and outstanding.

Despite these achievements, the problem of the structure of hadrons has remained elusive and controversial, to the point that no conclusive model of hadron structure can be claimed at this moment. Physics is a discipline with an absolute standard of values: the physical veritas. Until theoretical ideas (such as the hypothesis that quarks are physical particles) are experimentally established on unequivocal grounds, they constitute conjectutes or beliefs, but not the manifestation of physical truth. This is the reason for the use of the term "quark conjectures" in the title of this paper and in the following analysis.

The objective of this paper is to present an outline of the criticisms of quark models which have been present in the community of basic research for some time. As we shall see, these criticisms are numerous, and most of them are of a delicate nature because they involve an assessment of the validity of our current theoretical knowledge when applied to the description of the strong interactions. The treatment of these issues is not an easy task. Therefore, I appeal to the understanding of the receptive reader in case my presentation is sometimes deficient in completeness and rigor.

[3] See Ref. 3 for a review of the early stages of quark models, Ref. 4 for a more recent account, Ref. 5 for a review of charmonium and related topics, and the series of reprint volumes[(6)] for the annual progress in the field.

[4] It should be indicated here that Lagrange's equations in the form conventionally used in QCD and in the contemporary physics literature

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathscr{L}}{\partial \phi^{k}{}_{;\mu}}\frac{\partial \mathscr{L}}{\partial \phi^k} = 0; \qquad \phi^{k}{}_{;\mu} = \frac{\partial \phi^k}{\partial x^\mu}; \qquad k = 1, 2, \ldots, N;\ \mu = 0, 1, 2, 3$$

are *erroneous* whenever the symbol $\partial/\partial x^\mu$ is interpreted in its conventional mathematical meaning, that of representing a *partial* derivative. The correct form of the equation is that with a *total* derivative,[(7)] i.e.,

$$\frac{d}{dx^\mu}\frac{\partial \mathscr{L}}{\partial \phi^{k}{}_{;\mu}} - \frac{\partial \mathscr{L}}{\partial \phi^k} = \frac{1}{2}\left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{k}{}_{;\mu}\,\partial \phi^{i}{}_{;\nu}} + \frac{\partial^2 \mathscr{L}}{\partial \phi^{k}{}_{;\nu}\,\partial \phi^{i}{}_{;\mu}}\right)\phi^{i;\mu}{}_{\nu}$$
$$+ \frac{\partial^2 \mathscr{L}}{\partial \phi^{k}{}_{;\mu}\,\partial \phi^{i}}\phi^{i}{}_{;\mu} + \frac{\partial}{\partial x^\mu}\frac{\partial \mathscr{L}}{\partial \phi^{k}{}_{;\mu}} - \frac{\partial \mathscr{L}}{\partial \phi^k} = 0$$

In any case, my hope is that these criticisms will be considered by experts on quark models and that countercriticisms will be presented for a possible future resolution of this rather crucial aspect of basic research. Also, the current compartmentalized studies along quark and nonquark lines with little (if any) interplay is rather unappealing in terms of scientific effectiveness. It appears advisible that supporters and opponents of quark models join forces for an effective resolution of their differences, in the traditional pursuit of scientific knowledge.

As indicated earlier, the disagreements between supporters and opponents of quark models are varied and numerous ranging, from conceptual issues to differences in research attitudes and specific technical aspects. In this section we shall deal with preliminary issues only. Specific arguments will be considered in the subsequent sections.

The most visible problematic aspect of the quark models is that, despite rather large investments of financial and human resources through the years, the existence of quarks has not yet been experimentally established.

A first point of controversy is created by indications of the possible experimental detection of fractional charges.[8,9] This is generally (although not universally) seen by quark supporters as potential evidence of the possible existence of quarks in nature. Quark opponents disagree. The term "quarks" refers to a number of different particles carrying different quantum numbers. In order to establish unequivocally the quarks as the physical constituents of hadrons, *all* quarks must be *individually* established on clear experimental grounds. Besides epistemological reasons for the needed scientific rigor, there are technical reasons in support of this attitude. Indeed, as we shall indicate later, *fractional charges are allowed within a hadron by models fundamentally different from the quark models.*[10]

A second area of current disagreement is related to the so-called problem of confinement. Hadrons generally exhibit spontaneous decays. Since quarks are not produced free in these decays, and since their experimental detection in inelastic scatterings under all available (high and low) energies has remained inconclusive, a new trend has been initiated and lately intensified, consisting in the construction of a model whereby quarks, while being the physical constituents of hadrons, are confined within their charge volume and cannot be produced free. Despite brilliant contributions to a rather formidable theoretical problem, no conclusive model of confinement which can be accepted by the scientific community at large has been achieved.[11]

The controversies on the problem of confinement are numerous and range from epistemological to conceptual and technical issues. On epistemological grounds, the possible achievement of a final model of confinement is considered by some (but not all) quark supporters as virtually conclusive evidence of the validity of the quark models. Quark opponents disagree.

The idea is that the achievement of a conclusive model of confinement will simply shift the need of establishing on clear experimental grounds each and all quarks from outside to within a hadron. According to this view, the quark models will then remain fundamentally unestablished until a new technology capable of detecting particles under strong interactions is available. It is appropriate to point out here that the experimental capability available at this time for the direct detection of particles involves long-range electromagnetic interactions, e.g., as in bubble chambers. Predictably, a number of generations will pass until we are able to achieve the new technology indicated here of directly detecting particles under the short-range ($1\ F = 10^{-13}$ cm) strong interactions.

On conceptual grounds the disagreements are perhaps deeper. It is rather generally acknowledged that the constituents of hadrons are lightly bound. Since these bound states are unstable and exhibit spontaneous decays, the idea that they are confined meets with an understandable uneasiness. Indeed, one nonquark line of studies currently under investigation (which we shall indicate later) is based on the idea that, under the assumption that the constitutents are lightly bound and the states are (highly) unstable, the constituents can be produced free in spontaneous decays. In conclusion, there exists a considerable uneasiness in one segment of our community toward the very idea of confinement.

On technical grounds, the disagreements may be even deeper. They are due to the fundamental discipline used in quark models, quantum mechanics. Tunnel effects are a natural consequence of the very foundations of this discipline. The achievement of an unequivocal model of confinement therefore touches the foundations of this discipline, and calls for the construction of a model of an unstable bound state in which the probability of the tunnel effect is literally (and not approximately) zero. This does not appear to be an easy task within the context of the rather inflexible laws of quantum mechanics.

Another area of disagreement is due to the trend toward a progressive increase in the number of different quarks. There is no doubt that this trend has sound technical motivations within the context of quark models. Indeed, it is needed to properly accommodate new particles and their data, as made available by the experimenters or as predicted by theorists. Yet, an increase in the number of different, unidentified, quarks without a direct experimental backing is seen by quark opponents as an increase (rather than a decrease) of the problematic aspects of these models. Actually, the possibility that there exist infinitely different hadrons (and leptons) does not appear unrealistic at this moment. The concern is that such a situation might demand an infinite number of different quarks for these models to be consistent.[5] In conclusion,

[5] The efforts to limit this number to 30 should be indicated here. This point is controversial per se. As we shall elaborate in Sections 3 and 4, there appears to be a disagreement

whether quark models need a finite or an infinite number of quarks, the increase of the number of quarks has met with an understandable uneasiness and concern by one sigment of our community.

These introductory remarks are sufficient to indicate the delicate status of the studies of a rather fundamental physical problem. But these remarks are related to easily identifiable aspects only. As we shall see, the reasons for disagreement are of a much more substantial nature.

Before entering into these issues, it appears advisable to clarify one aspect of potential misunderstanding between quark supporters and opponents, which is apparently basic to the current lack of scientific cooperation between these groups. In the simplest possible terms, quark supporters often interpret criticisms of quark models as intended to invalidate the physical value of the unitary groups for hadrons. One of the objectives of this paper is to dispel this possible misunderstanding as clearly as possible.

According to a by now historical process, new insights in physics never "destroy" previous accomplishments of proved physical value. They simply identify their arena of applicability and implement them in a broader conceptual and technical context. For instance, Albert Einstein did not claim that Galileo Galilei was wrong in his relativity ideas. He simply identified the physical arena in which Galilei's relativity applies, and proposed a covering relativity for a broader physical arena.

We have stressed earlier in this section the brilliant achievements of unequivocal physical value of the unitary models in hadron physics. These results are now part of the history of pkysics and no possible future advances can invalidate them. Therefore, the issue of whether unitary models are valid or invalid has no scientific value.

A scientifically valuable problem, instead, is the identification of the part of the hadronic phenomenology to which the unitary models unequivocally apply, and the different or broader part in which they are unestablished and problematic, and for which the search for possible more general conceptual and technical approaches may be scientifically productive.

on the very definition of what a model of the structure of hadrons should be, and what are the data to be quantitatively represented. The issue referred to here is whether 15 quarks and 15 antiquarks are capable of representing *all* hadrons currently known and *all* hadrons expected to be identified in the future, as well as *all* their total characteristics and, last but not least, *all* their decay modes and *all* the related fractions. In conclusion, we are referring here to a rather substantial volume of data already existing and which is expected to multiply in the near and far future, which should be *all* quantitatively represented by 15 quarks and 15 antiquarks, according to quark supporters. Other researchers expect instead the need to multiply the number of different quarks in a measure proportional to the increase of data, in order to avoid inconsistencies.

A tentative answer is the following.[12]

(A) The unitary models are unequivocally valid for the Mendeleev-type, exterior, "chemical" classification of hadrons only.

(B) The quark models are conjectural at this time only when taken as constituting a structure model for each individual element of a given unitary hadronic multiplet.

(C) In relation to the open problem of hadronic structure, studies using quark conjectures should be continued. Studies using fundamentally different conjectures should be continued also, under the condition that they achieve full compatibility with the established unitary models of Mendeleev-type classification.

Aspect (A) is self-evident. A simple inspection of the diagrams of Fig. 1 indicates that the validity of the unitary models for hadronic classification is simply incontrovertible. In Ref. 12 the unitary models were therefore assumed as the final classification of hadrons.

Aspect (B) is not equally self-evident. As a matter of fact, it may appear contradictory to aspect (A) under a first inspection. This second aspect emerged after a comparative anamnesis of the current status of the art in hadron physics with the historical solution of the problem of atomic phenomenology.

Atomic phenomenology demanded two different, yet compatible models, one of classification (Mendeleev) and one of structure (Bohr–Thomas–Fermi). These two models are profoundly different in conception, methods, and insights. Yet they are compatible, in the sense that the model of classification produced invaluable elements for the structure, while the structure model recovered the classification.

At the atomic level, the idea that one single model could serve for both the classification and structure of atoms was extraneous to the analytic mind of the founding fathers of contemporary physics.

At the hadronic level, the situation does not appear to be necessarily the same. Even though not explicitly stated in the available literature, quark models are intended to serve for both the classification *and* structure of the hadrons. A reinspection of the diagrams of Fig. 1 with tensorial products (1) and the quark hypothesis establishes this dual *intent* of the quark models of providing the classification of hadrons into unitary multiplets, and, jointly, the structure of each individual element of a unitary multiplet.

Points (A) and (B) are intended to stress the fact that, while the physical value of the unitary models of classification is unequivocal, the character of these models as also providing the structure of hadrons is conjectural at this time. The idea was to identify the arena in which unitary models are unequivocal and the different arena in which they are conjectural, in the hope of minimizing or otherwise avoiding unnecessary controversies. Apparently, this identification had not been considered in the (rather vast) literature prior to Ref. 12.

Point (C) was inspired by the traditional pursuit of open physical problems. There is no a priori reason available at this time to expect that the dichotomy classification/structure which was necessary at the atomic level will eventually be necessary also for the hadronic phenomenology. Thus, studies along the line of quark conjectures are encouraged. But there is equally no a priori reason that the hadronic phenomenology should not eventually demand (at least) two different, yet compatible models, one of classification and one of structure, in exactly the same way as occurred for atoms. Thus, the studies of models of structure fundamentally different than those of quark inspiration should be continued also. The need for these latter models to recover the unitary classification (which is unnecessary for the quark models) is then self-evident.

It is hoped that the reader can see the lack of contradictions in point (C) per se, as well as in relation to (A) and (B). These reason is quite simple. If experiments eventually rule out the existence of quarks as physical particles,[6] this will leave the physical validity and consistency of the unitary models of classification completely unaffected. In fact, at the level of classification, a quark is a mere mathematical quantity: *a representation of a Lie group*. It becomes a *particle* only when the same models are implemented to represent the totality of the hadronic phenomenology (classification, *and* structure, *and* scattering).

It is hoped that these remarks provide final reassurance to quark supporters that no physicist intends to (or actually can) invalidate the achievements of the unitary models. At the same time, it is hoped that this paper will provide arguments to stress that the restriction of all research on the fundamental problem of hadron structure to only quark conjectures may turn out to be a historical error.

We now present the main arguments of doubt about quarks, with the strict understanding that *the following parts of this paper solely refer to the interpretation of quark models as structure models of hadrons*. Under no circumstances are they intended to refer to the unitary classification of hadrons. For this reason we shall use the term "unitary models" when referring to the Mendeleev-type classification of hadrons, and the term "quark models" when they are assumed to constitute structure models.

[6] As we shall see, this possibility is considerably greater than is conventionally acknowledged in the quark literature, provided that the experimenter considers on equal grounds feasible experimental proposals for either the proof or the disproof of the quark conjectures.

Also, we shall use the term "quark models" rather than "quark model," due to the current lack of selection of one single model among a rather considerable variety of different models.[13]

As a final comment, it should be clearly indicated that, to avoid prohibitive length, this paper is not intended to be a technical treatment of the new, Bohr-type structure model of hadrons[12] (which is not of quark inspiration). In fact, these studies are rather numerous and rapidly expanding,[7] to the point of rendering prohibitive a technical treatment in a paper devoted to a related, but ultimately different issue.

In summary, this paper is devoted to an outline of the numerous problematic aspects of the conjecture that quarks are physical particles, under the assumption that the unitary models provide a final Mendeleev-type classification of hadrons. The paper will present only incidental comments on the new, Bohr-type structure model of hadrons, which is incompatible with quark conjectures, yet is capable of achieving compatibility with the established unitary models of classification.[12,14–16]

2. THE FUNDAMENTAL ASSUMPTIONS OF THE QUARK MODELS

The issue of whether quarks exist or not is, in the final analysis, of rather limited physical depth. A more scientifically worthwhile issue is the classification of all the assumptions of quark models and the identification of their interrelations, where the term "assumptions" indicates hypotheses, laws, and disciplines lacking a clear and unequivocal experimental backing in the physical context to which they refer.

This broader approach to the problem is clearly essential, first, to avoid unnecessary controversies, and, second, to initiate a systematic theoretical study for the formulation of a series of experiments to determine whether the quark models do indeed constitute the true model of hadron structure or not.

We begin in this way to enter into the actual topics of disagreement. The idea is that, until the basic assumptions of the quark models have been experimentally established on clear grounds, the search for quarks may be sterile. Indeed, it is possible that the basic assumptions of the quark models

[7] The new model of hadron structure here referred to is based on the so-called Lie-admissible approach to strong interactions, and will be indicated in Sections 3 and 4. For more detailed treatment the interested reader is referred to the review memoir[14] and to the proceedings of the workshops on Lie-admissible formulations.[15] The series[14] reprints all mathematical, theoretical, and experimental papers directly or indirectly related to the model.

can be invalidated by experiments (see the subsequent sections), in which case the search for quarks would turn out to be meaningless.

This situation is created by the fact that, at a deeper analysis, the quark models are based on a rather complex "topology" of assumptions, each one of rather fundamental physical character, but without experimental backing at this time. As a matter of fact, the assumption that quarks are the constituents of hadrons is of secondary nature with respect to other emerging assumptions.

A classification of the basic assumptions of quark models and a study of their relationships has been conducted by in Ref. 12 (see Refs. 17 for a detailed treatment and Refs. 18 for basic tools). An outline of the results of these studies is presented below with the understanding that it is tentative and in need of assessment by independent researchers.

The assumptions of quark models can be classified into two groups, here called "primary assumptions" and "secondary assumptions." This classification is intended to identify the assumptions of primary physical character and those of derived character only, that is, consequential to the primary assumptions. In this way, the classification should be effective for the identification of the interrelationships among different assumptions.

The primary assumptions of quark models are the following.

Primary Assumption 1. Einstein's special relativity, as currently known, is valid for the hadronic constituents (or structure) as well as, more generally, for the strong interactions.

This is not the place to recall the unequivocal validity of Einstein's special relativity for electromagnetic interactions, as proved, say, in particle accelerators. Nevertheless, the validity of the same relativity under strong interactions is a mere conjecture at this time, lacking direct and incontrovertible experimental evidence. In fact, a detailed analysis of the structure of current experiments in high-energy physics[14] clearly indicates that special relativity is merely *assumed* in the data elaborations. Since these experiments are not intended to test the relativity considered but instead are devoted to other topics (identification of new particles, measurement of their physical properties, etc.), the results of currently available experiments cannot provide final evidence on the validity of Einstein's special relativity under strong interactions. At best, they can provide *plausibility arguments*. We reach in this way a first fundamental open problem of quark models as well as of contemporary physics at large. Einstein's special relativity is essential for the consistent definition of quarks. Jointly with the continuation of current experimental trends, it therefore appears advisable to establish the validity of this relativity in the arena of interest via direct and conclusive experiments. The formulation of these experiments is one of the primary objectives of the Lie-admissible approach to strong interactions[12,14–16] As we shall review in Section 3, there exist a number of arguments indicating a conceivable inapplicability of Einstein's special relativity for the strong interactions, and the need of generalized relativity specifically conceived for the arena considered. If this possibility is eventually established by experiments, the problem of the structure of hadrons must be clearly reinspected ab initio (Section 4).

Primary Assumption 2. Quantum mechanics, in its currently available, nonrelativistic, relativistic, and field-theoretic versions, is valid for the hadronic constituents, as well as, more generally, for the strong interactions.

This is also not the place to recall the unequivocal validity of quantum mechanics for atomic structure, as well as for electromagnetic interactions. Nevertheless, this is not sufficient to establish the universal validity of the discipline for the entire microscopic world, irrespective of the physical conditions of the particles. An inspection of the currently available experiments indicates again[14] that the basic laws and principles of quantum mechanics (Pauli's exclusion principle, Heisenberg's indeterminacy principle, etc.) are merely *assumed* as valid in the data elaborations of current experiments in strong (nuclear and high-energy) interactions. On grounds of necessary scientific caution, we must therefore identify quantum mechanics as being conjectural for hadronic structure. As a matter of fact, a number of arguments exist leading to a conceivable inapplicability of quantum mechanics (in an exact form) for the strong interactions, and to the need of a covering discipline specifically conceived for the arena at hand (Section 4). Again, these remarks should not be interpreted as denying the *plausibility* of quantum mechanics for the strong interactions. They are merely intended to identify a second fundamental open problem of quark conjectures and of contemporary physics at large. It is evident that, in case the basic laws of quantum mechanics need suitable generalizations to represent particles under strong interactions, the problem of the structure of hadrons must also be reinspected ab initio, and the hypothesis that quarks are physical particles becomes a mere historical episode (Section 4).[8] The formulation of experiments showing the validity or invalidity (exact or approximate validity) of quantum mechanics under strong interactions is another basic objective of Lie-admissible studies.[12,14–18] The primary arguments will be reviewed in Sections 3–5. Some authoritative, historical voices of doubt will be recalled in Section 6.

[8] Due to the deep impact of any form of relativity for the (classical and) quantum mechanical description of nature, a conceivable inapplicability of Einstein's special (and Galilei's) relativity for the strong interactions will inevitably call for a revision of quantum mechanics. Conversely, a generalization of conventional quantum mechanical laws will call for a revision of the underlying relativity. As a result, the primary assumptions 1 and 2 are, in the final analysis, deeply interrelated.

Primary Assumption 3. The quarks (and the antiquarks) are the elemental constituents of hadrons.

It is hoped that the remarks of Section 1 are sufficient to identify the current conjectural character of quarks as the constituents of hadrons. By and large, this assumption is acknowledged in the current literature. The apparent novelty of the analysis of Ref. 12 is the identification of the fact that this assumption, even though independent from assumptions 1 and 2, carries a physical weight smaller than those of the preceding assumptions. In other words, the fact that the quark models are based on the conjecture of the validity of the basic relativity and quantum mechanical disciplines for the hadronic structure is of a much more fundamental character than the assumption that quarks are the constituents of hadrons. As a matter of fact, it was submitted that perhaps the current problematic aspects of quark models are only the symptoms of a conceivably much more fundamental problem of consistency at the level of the basic laws.[12,14,17]

Each of the primary assumptions indicated above possesses a variety of secondary assumptions of derived character. Without any claim of completeness, we quote the following ones.

Secondary Assumption 1a. The notion of constituent of atomic structure identically applies to hadronic structure, apart from unitary implementations.

A truly crucial prerequisite to properly formulate the problem of structure, let alone to properly treat it, is to achieve a quantitative characterization of the notion of constituent particles. A "particle" is nowadays technically identified via (representations of) the applicable relativity (Galilei's or Einstein's special relativity). This (nonrelativistic or relativistic) notion of particle is incontrovertible when it refers to atomic structure or to electromagnetic interactions. Nevertheless, the issue of whether the same notion identically applies to hadronic constituents and to strong interactions in general (apart from the additional unitary internal degrees of freedom) appears to be fundamentally open at this time on both theoretical grounds (Sections 3 and 4) as well as experimental grounds (Section 5). Clearly, this situation is a consequence of the primary assumption 1.

Secondary Assumption 1b. The familiar linear, local, Lorentz covariance law of fields, Eq. (3), applies to the fields representative of the hadronic constituents or of particles under strong interactions.

As we shall see, despite the unequivocal validity of this law for electromagnetic interactions, the validity of the same law for strong interactions is questionable on rather numerous grounds. In any case, law (3) does not possess a clear, direct, and unequivocal experimental backing. As such, it must be considered as of conjectural character at this time, on grounds of necessary

scientific caution. Again, this situation is a consequence of the primary assumption 1. The relationship with secondary assumption 1a is self-evident.

Secondary Assumption 2a. Pauli's exclusion principle and the spin-statistics theorem are valid for the hadronic constituents as well as, more generally, for particles under strong interactions.

An inspection of the original (by now historic) papers on this subject[19] reveals that Pauli's principle was specifically conceived for atomic structure, in which it subsequently proved to be essential for the representation of a number of fundamental aspects of the Mendeleev classification (this is an aspect of the interplay between the model of classification and that of structure at the atomic level). In more recent times, the principle has been applied to nuclear structures, with results in excellent agreement with experimental data. Nevertheless, while the experimental backing of the principle at the atomic level is simply incontrovertible, the situation does not appear to be the same at the nuclear level. The experimental verification of Pauli's exclusion principle in nuclear physics was proposed[12] to establish whether the principle is valid in nuclear physics in the same quantitative measure as in atomic physics, or whether very small deviations are detectable. In different terms, the question which was submitted in Ref. 12 is whether our current knowledge of the validity of Pauli's principle in nuclear physics is quantitatively comparable to current knowledge of the validity of the PCT symmetry in particle physics, or is at a stage prior to the discovery of, say, parity violation in weak interactions. It is understood that, due to the good agreement with experimental data, possible departures from Pauli's principle in nuclear physics can at most be very small. The situation at the hadronic level appears to be different due to the rather profound physical differences of these two layers of microscopic reality. In any case, Pauli's principle for the hadronic constituents is a *conjecture* at this moment, without direct experimental backing; that is, aside from indirect experimental indications based on a variety of different assumptions. Again, these remarks are not intended to deny the plausibility of Pauli's principle in hadron physics. Instead, they are intended to stimulate the experimental resolution of the issue, as the only basis for the conduct of sound physical studies. The spin-statistics theorem is, to a considerable extent, the field-theoretic image of Pauli's principle in discrete quantum mechanics, and its lack of final experimental verification for strong interactions can also be seen by inspecting available experimental data. The derivative character of the assumption under consideration from the primary assumption 2 is self-evident.

Secondary Assumption 2b. Canonical quantization rules, in their nonrelativistic, relativistic, and field-theoretic versions, apply to hadronic structure and, more generally, to strong interactions.

Again, the validity of these rules for the electromagnetic interactions is established. Nevertheless, the validity of the same rules for the strong interactions is not equally clear and a number of arguments indicating their conceivable invalidity will be indicated in Section 4. The implications of this assumption are rather deep, and so are its relationships with the other assumptions under consideration. Indeed, if these rules are invalid for the strong interactions, the Lie algebra realization of the basic relativity laws in terms of quantum mechanical operators is invalid. The deep compatibility and inflexible relationship of the various aspects of contemporary theoretical physics begins to emerge. Pauli's exclusion principle, the spin-statistics theorem, the Lorentz covariance law, the notion of particle, and numerous other insights (e.g., Heisenberg's indeterminacy principle) are all links of one, single, rigid chain of mutually compatible insights. The possibile invalidity of only one link of this chain may imply the invalidity of the entire construction. A similar argument applies for the possible validity. This remark is made in the hope that it can assist the experimenter in the selection of that link of the chain whose validity or invalidity can be more readily established for the strong interactions in a way independent from its validity for electromagnetic interactions.

Secondary Assumption 3a. Quarks are composite constituents of hadrons.

As indicated by recent studies[8] the question of whether the hadronic constituents (for the case of heavy or superheavy hadrons) cluster into granules with fractional charges or not does not appear to be readily resolvable on the basis of our knowledge at this time. Therefore, it appears advisable to differentiate primary assumption 3 into a secondary assumption in which quarks are assumed as composite. In different terms, the problem of whether quarks exist or not is in actuality of multifold nature. First, there is the problem of whether the quarks are the final, elemental, indivisible constituents of hadrons. If experiments disprove this conjecture, there still remains the possibility that quarks are composite constituents of hadrons. As we shall see, a crucial function for the possible resolution of this issue is given by the problem of the structure of the light hadrons. This is why the problem of the structure of the octet of light mesons of Fig. 1 was considered to be of fundamental character in Ref. 12, and the same attitude is preserved in this paper.

Secondary Assumption 3b. Quarks are confined.

It is hoped that the remarks of Section 1 are sufficient to indicate the

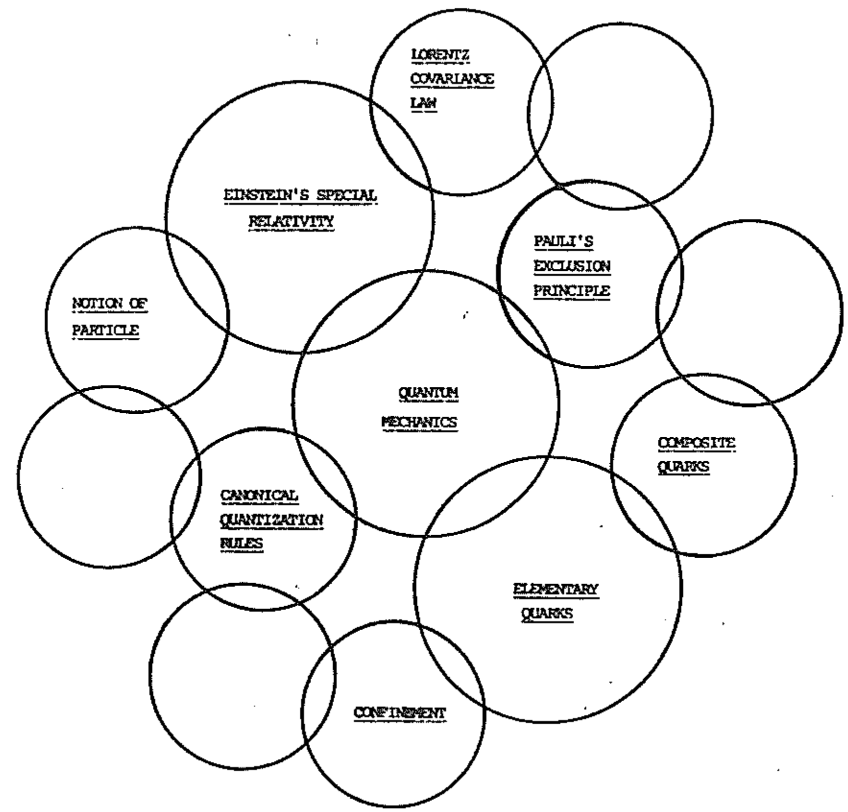

Fig. 2. The rather complex "topology" of assumptions of the quark models, schematically presented via "overlapping polycircles." According to the analysis of Ref. 12, quark models and QCD, when interpreted as dealing with the structure of hadrons, are based on rather numerous assumptions of fundamental physical character, all lacking a clear, direct, and unequivocal experimental backing. The experimental search for quarks then becomes of secondary relevance, when related to the need for a prior experimental resolution of the validity or invalidity of the basic physical laws of the models considered in the arena considered. The analysis is not intended to deny the plausibility of the laws considered in the arena considered, but merely intended to stimulate the experimental resolution, in due time, of these fundamental problems. Undoubtedly, the emergence of the conjectural character of the basic physical laws of quark models of hadron structure casts shadows on the hypothesis that the quarks are the constituents of hadrons and, in any case, calls for a comprehensive study of the formulation and conduct of a variety of experiments to eventually resolve the validity or invalidity of these models of hadron structure. This situation has been lingering in the community of basic research for some time. The reader may be amused to know that quarks have been referred to in the literature as the "Yeti particles," or that the case of quarks has been amiably related to the historical case of the ether, or that the role of the (rather powerful) quark supporters in contemporary theoretical physics has been compared to the control of science by priests during Galilei's time.

need of considering this an assumption at this time on grounds of scientific caution.

We have listed here only some of the most significant assumptions of the quark models which are implied by the primary assumptions 1–3. The reader can easily complete this classification with all additional aspects considered valuable.

The attitude implemented in this section should be indicated. It essentially consists in a critical examination of each and every methodological tool used in the actual realization of the quark models. The objective is that of identifying those tools that are unequivocally valid (e.g., a Fourier transform) and those that are conjectural (e.g., Feynman diagrams[9]) as a necessary prerequisite for the sound conduct and presentation of research.

A summary view of the content of this section is provided in Fig. 2.

3. THE PROBLEM OF THE ARENAS OF VALIDITY AND INVALIDITY OF EINSTEIN'S SPECIAL RELATIVITY IN PARTICLE PHYSICS

The belief that Einstein's special relativity, as currently known, constitutes the final form of relativity applicable to particle physics, with unlimited applicability to all conceivable physical conditions of particles, has no scientific value.

A more scientifically productive issue is the identification of the arena of particle physics in which Einstein's special relativity unequivocally applies, and the different or broader arena in which it is a conjecture at this time, and for which a covering relativity becomes conceivable.

This latter issue is scientifically productive on more than one ground. First, the identification of the arena of conjectural character of the relativity considered is a prerequisite for the subsequent problem of formulating and conducting experiments aiming at the verification of its validity or invalidity. Second, and equally important, the identification of the arena of conjectural validity of the relativity considered is a prerequisite for stimulating studies on a conceivable covering relativity. Predictably, these studies are expected to involve the identification of new mathematical tools. Whether such a conceivable covering relativity can be consistently constructed or not, and whether it will conform to physical reality or not, studies of this nature can only be valuable for the advancement of mathematical and physical knowledge.

[9] As we shall see, Feynman diagrams are conjectural for the strong interactions because (unlike the case of the electromagnetic interactions) the Lagrangian representations are conjectural for the interactions considered.

The issue under consideration has been studied in detail in Refs. 12, 14, 17, 18, and 20. The answer which has been submitted is the following.

1. The arena of unequivocal validity of Einstein's special relativity in particle physics is that of the electromagnetic interactions only.[10]

2. An arena of current conjectural character of the relativity considered in particle physics is that of the strong interactions in general and of the strong hadronic forces in particular.

3. Studies based on the validity of the relativity considered for strong interactions and for the structure of hadrons should be continued. Studies based on the invalidity of the relativity considered in this latter arena should also be continued, jointly with the search for a possible covering relativity specifically conceived for the strong interactions.

The argument in favor of the applicability of Einstein's special relativity for the strong interactions is known, and it is provided by the quark models, as well as QCD.[6] It may be of some value for the interested reader to briefly outline here the main arguments of Refs. 12 and 14, which suggest a possible invalidity of the relativity considered in the arena of interest.

The folloling are well-established experimental facts.

I. The "size" (charge radius) of hadrons does not increase appreciably with mass (contrary to the corresponding occurrence at the nuclear level), and it is of the order of 1 F for all hadrons.

II. The size of hadrons coincides with the range of the strong interactions.

III. Hadrons are constituted by wave packets.

A picture of the strong interactions profoundly different than that of the electromagnetic interactions results from these experimental data. In essence, the latter interactions are long range, and thus can occur at distances much greater than the charge radius of the particles, as is typically the case for atomic structure. Under these conditions, the particles can be effectively approximated as being pointlike (Fig. 3). This approximation has several important consequences. First, (long-range) *electromagnetic interactions admit an effective treatment via local models*. The latter terms refer, physically, to interactions occurring among a collection of isolated points and, mathematically, to interactions admitting a representation via ordinary or partial differential equations. Second, pointlike particles can only experience "action at a distance" forces, that is (local) forces satisfying the integrability conditions

[10] One may add also the weak interactions. We have excluded them from point 1 because, in our view, the problem demands a detailed study (rather than an a prioristic assumption).

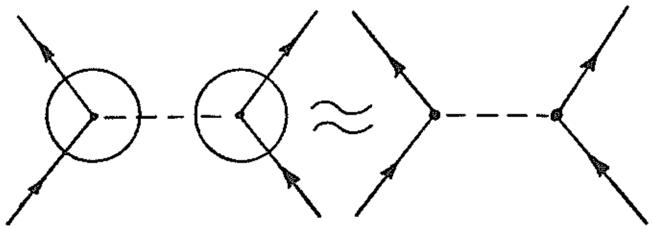

Fig. 3

for the existence of a potential function (the so-called conditions of variational self-adjointness; see Appendix A). As a result, *the electromagnetic interactions admit an effective treatment via (local) Lagrangian or Hamiltonian models.* Third, the Hamiltonian character of the equations of motion implies the applicability of conventional (Lie) algebraic and (symplectic) geometrical methods. As a result, *Lie algebra and symplectic geometry are applicable for the quantitative treatment of electromagnetic interactions.* The applicability of Einstein's special (or Galilei's) relativity then results. It should be stressed here that the entire physical and mathematical edifice rests on the capability, under the conditions considered, to approximate the particles as being massive points, according to the notion originally conceived by Galilei and Newton, and subsequently embraced by Einstein.

In the transition to the strong interactions the virtual entirety of the physical and mathematical treatment of the electromagnetic interaction appears questionable on a number of grounds. Experimental facts I and II establish that, as a necessary condition to activate the strong interactions, hadrons enter into a state of mutual penetration or overlapping of their charge volumes (Fig. 4a). As a consequence, *particles cannot be effectively approximated as being pointlike.* Furthermore, experimental fact III establishes that hadrons under strong interactions actually consist of wave packets under a condition of mutual overlapping (Fig. 4b).

The inappropriateness of the pointlike approximation of particles has a number of important consequences. First, it implies that the *strong interactions are expected to demand a suitable form of nonlocal model.* This form can be identified via experimental fact III. It is known that, in classical continuum mechanics, overlapping waves demand integrodifferential equations, that is, equations possessing a local (ordinary or partial) component, and an integral component representing the interactions at *all* points of overlapping. Apart from problems of quantization, there is no reason to expect that overlapping quantum waves can be treated via differential equations without the integral component.

The term "nonlocal models" suggested by experimental facts I–III can

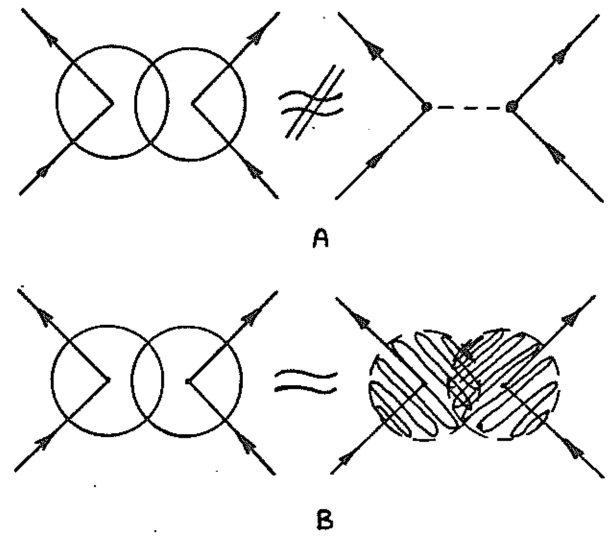

Fig. 4

now be made more precise. It refers, physically, to interactions occurring at all points of the volume of wave overlapping, and, mathematically, to interactions admitting a representation via ordinary or partial integrodifferential equations. The local-differential part is expected to represent the electromagnetic interactions as well as a component of the strong interactions (e.g., as in nuclear physics). The nonlocal-integral part is intended to represent the effect resulting from wave overlapping.

A further, rather crucial point that needs to be made precise concerns the nature of the nonlocal term. Contemporary high-energy physics is virtually dominated by Lagrangian (or Hamiltonian) models. This situation, however, is not sufficient to warrant that all interactions must be of Lagrangian or Hamiltonian type. In fact, it is well established in continuum mechanics that nonlocal forces resulting from wave overlapping are, in general, of nonpotential (i.e., non-Lagrangian or non-Hamiltonian) type. Again, there is no a priori reason to expect that, in the transition to quantum overlapping waves, this analytic character should be changed.

The alternative (potential or nonpotential character of the nonlocal forces) can be better illustrated in the arena where our intuition is surest, Newtonian mechanics. When a (local) Newtonian system admits only potential forces, the constituents are pointlike and admit only action at a distance. In order to represent the extended character of the objects responsible for contact (e.g., frictional) effects, the forces are selected to be of nonpotential type. We can equivalently say that the *nonpotential* (nonself-adjoint; see Appendix A) character of the nonconservative forces is representative of the *extended structure* of the objects considered. In the transition to classical overlapping waves we have an equivalent situation. In fact, the nonpotential character of the nonlocal forces is representative of the extended character of the waves as well as of the continuum generalization of the contact forces of Newtonian mechanics. In the transition to quantum overlapping waves, gain, there is no a priori reason to expect that this physical characteristic, so deeply related to the actual representation of particles as extended objects, should be abandoned, and more simplistic equations should be assumed.

We reach in this way the representation of the strong interactions via nonlocal (integrodifferential) field equations as suggested by experimental facts I, II, and III and as proposed in Ref. 12:

$$\{[(\Box + m_{(k)})\phi_k - f_k(\phi, \phi^i{}_\mu)]_{SA} - \int d^4x\, F_k(x, \phi, \phi^i{}_\mu)\}_{NSA} = 0 \tag{4}$$

$$K = 1, 2, \ldots, N; \quad \mu = 0, 1, 2, 3$$

with Newtonian image[11]

$$\left\{[m_{(k)}\ddot{\mathbf{r}}_k - f_k(t, \mathbf{r}, \dot{\mathbf{r}})]_{SA} - \int_V d\mathbf{r}'\, F_k(t, \mathbf{r}, \mathbf{r}', \dot{\mathbf{r}}, \dot{\mathbf{r}}')\right\}_{NSA} = 0 \tag{5}$$

where SA stands for self-adjointness (existence of a Lagrangian) and NSA stands for nonself-adjoiness (lack of general existence of a Lagrangian) (Appendix A).

The argument of Ref. 12 is that *Einstein's special relativity becomes incompatible with the strong interactions when represented via nonlocal forces nonderivable from a potential.*

A study of this intriguing situation has indicated that the breakdown of Einstein's special relativity for systems of type (4) in actuality occurs at the level of the mathematical foundations of relativity.

At the analytic level there is the general impossibility of even defining the canonical formalism, trivially, because of the general lack of existence of a Hamiltonian. Thus, the entire conventional analytic context (including its canonical part) is in question for the strong interactions already at the classical level, let alone the quantum mechanical level. It is hoped that the reader can

[11] The following formulation also can be considered:

$$\left\{[m_{(k)}\ddot{\mathbf{r}}_k - \mathbf{f}_k(t, \mathbf{r}, \dot{\mathbf{r}})]_{SA} - \int_V d\mathbf{r}\, \mathbf{K}_k(t, \mathbf{r}, \dot{\mathbf{r}})\right\}_{NSA} = 0$$

begin to see why the canonical quantization rules have been identified as a conjecture in Fig. 2, although the true technical situation will appear at the quantum mechanical level (Section 4).

At the algebraic level there is a general incompatibility of the interactions considered with Lie's theory as currently known, in the sense that the Lie algebras cannot be introduced via the brackets of the Hamiltonian time evolution law. Thus, for the systems considered, the applicability of *all* Lie algebras is in question, let alone the Lie algebra of the Poincaré group.

At the geometrical level there is an incompatibility of the systems considered with the symplectic or contact geometry as currently known, trivially, because the realizations of this geometry in the admissible charts demand local differential equations.

These breakdowns have been indicated in the hope of preventing a simplistic attitude toward making Einstein's special relativity compatibility with systems of type (4). Before such an attitude can acquire scientific value, the interested researcher must first solve a rather substantial array of problems of compatibility of the systems considered with the mathematical foundations of the relativity considered.

In actuality, the inapplicability of Einstein's special relativity for the strong interactions is seen to be rather natural when the *experimentally established* conditions of Fig. 4 are inspected without prejudice. As clearly expressed by Einstein in his limpid writings, his relativity was conceived for pointlike charged particles moving in vacuum (no contact forces!) under an external, long-range electromagnetic field. There is no reason to expect that the relativity that is so effective for such a physical arena must necessarily apply to a fundamentally different physical context. After all, the strong interactions were unknown in 1905! When the fundamental character of the pointlike nature of particles in special relativity is identified, the need for a possible covering relativity for extended particles in conditions of mutual penetration becomes rather natural. Here the term "covering relativity" is intended to express the compatibility condition that a possible generalized relativity must recover special relativity at the limit when all particles are approximated as being pointlike, with the resulting null value of all nonlocal or local nonself-adjoint forces.

A first direct consequence of experimental facts I, II, and III is that the *conventional Lorentz transformation law of fields, Eq. (4), is expected to be incompatible with the strong interactions, first, because it is a local transformation law, and, second, because it is linear. If locality is preserved, at least the linearity should be abandoned to avoid excessive approximations.* In short, the best research attitude is that of searching for a generalization of special relativity and of the transformation law of fields specifically conceived for particles in the experimentally established conditions of Fig. 4.

To initiate this predictably laborious and long-term research project, the attitude of Refs. 7, 12, 14, 17, 18, and 20 has been as pragmatic as possible. It is a fact that virtually all of the methods of contemporary theoretical physics are strictly local in character. The direct transition to nonlocal settings appears to face rather substantial technical difficulties, assuming that they can be treated with existing mathematical and physical knowledge. The attitude advocated was therefore that of approximating nonlocal strong interactions with local models, but in such a way as to account for the extended size of the particles. The answer was provided by knowledge well established in Newtonian mechanics. It is known that nonlocal forces are well approximated in mechanics via local (nonlinear) forces nonderivable from a potential (e.g., via polynomial expansions in the velocities) yielding the Newtonian systems of everyday experience, such as damped oscillators, spinning tops with drag torques, trajectory problems in the atmosphere, etc. All these systems belong to the class of local, variationally nonself-adjoint systems

$$\{[m_k\ddot{\mathbf{r}}_k - f_k(t, \mathbf{r}, \dot{\mathbf{r}})]_{SA} - F_k(t, \mathbf{r}, \dot{\mathbf{r}})\}_{NSA} = 0 \tag{6}$$

The nonself-adjoint forces then represent precisely the motion of extended objects in a resistive medium, while the self-adjoint forces represent the action-at-a-distance forces.

Of course, before reaching a position where one can even partially confront the problem of the relativity laws, one has to identify mathematical methods for the treatment of systems of type (6), *at the pure classical level first*. The problem of quantization is only second.

The study of these methods has identified the existence of two different, yet compatible generalizations of Lie's theory, tentatively called:

(A) *Lie-isotopic generalization*, in the (algebraic) sense that the theory is still Lie in character, although the product is the most general possible (regular) form of the Lie algebra product.

(B) *Lie-admissible generalization*, in the (algebraic) sense that the theory is no longer Lie in character. Instead, it is of the covering Lie-admissible[12] character.

Both generalizations of Lie's theory admit a classical (and quantum mechanical) realization, based on corresponding generalizations of Hamilton's (Heisenberg's) equations. The classical analytic equations characterized by the Lie-isotopic generalization of Lie's theory were called *Birkhoff's equations*[18,20] for historical reasons. The classical analytic equations characterized by the Lie-admissible generalization of Lie's theory turn out to be the equations originally conceived by Hamilton, those *with* external terms, although written in an algebraically adequate form (see below). The corresponding quantum mechanical versions of these equations were apparently proposed for the first time in Ref. 12 (see next section).

Both generalizations of Lie's theory are "directly universal" for systems (6), that is, applicable to *all* systems of the class admitted (universality) without redefinition of the coordinates of the experimental detection system (direct universality). But only the Lie-admissible generalization of Lie's theory is directly universal for the most general systems known at this time, the variationally nonself-adjoint integrodifferential systems.

By using the methods available, we proposed (20) a *generalization of Galilei's relativity* for systems (6) which is of Lie-admissible algebraic character. A corresponding quantum mechanical generalization was proposed in Ref. 12. The studies for relativistic (classical and quantum mechanical) extensions are in progress.

Furthermore, by using the broader (bimodular) representation theory of the Lie-admissible algebras of operators, we proposed[12,21] a *generalized notion of particles under strong interactions*, that is, of extended particles under conditions of mutual penetration and overlapping of the wave packets. Finally, this generalized notion of particle was used to propose a *new Lie-admissible model of the structure of hadrons*[12,17] in which the constituents are physical particles that can be produced free under spontaneous decays.

A technical review of these studies would be impossible in this paper. Nevertheless, for the reader's convenience, as well as for subsequent notational needs, it is advisable to outline the truly essential ideas.

3.1. The Main Idea of the Birkhoffian Generalization of Hamiltonian Mechanics[18,20]

When studying systems of type (6) a first step is the identification of the integrability conditions for the existence of a Lagrangian, or, independently,[13] of a Hamiltonian representation. These are the conditions of variational self-adjointness indicated earlier. The second step is the identification of the methods for the construction of a Lagrangian, or, independently, of a Hamiltonian from the equations of motion under the integrability conditions. The third step is the identification of the property that the Lagrange and Hamilton equations of contemporary literature (sometimes called the

[12] A *Lie-admissible algebra* U is an algebra with elements $a, b, c, \ldots$ and (abstract) product ab over a field F (assumed of characteristic zero throughout this paper) such that the attached algebra U^-, which is the vector space U equipped with the product $[a, b]_U = ab - ba$, is a Lie algebra. When ab is associative, U is a conventional Lie algebra. Thus, the notion of associative Lie-admissibility is at the foundation of Lie's theory in its conventional form. When ab is nonassociative, we have the Lie-admissible generalization of Lie's theory studied in Refs. 12, 14–17, 20, 21, 23–33, and 42–46.

[13] That is, without a prior knowledge of a Lagrangian.

"truncated equations"[14]) *are not* directly universal in Newtonian mechanics. In fact, they can represent only a subclass of Newtonian systems in the coordinates of the experimental detection system (Appendix B).

A generalization of Hamiltonian mechanics is clearly necessary to avoid perpetual-motion-type approximations and to achieve the (much needed) direct universality.[15] The generalized mechanics identified by the conditions of variational self-adjointness has the following most salient properties.

Analytic Profile. Second-order systems can be turned into equivalent first-order systems (vector fields) via generally noncanonical prescriptions for the characterization of new independent variables. By using the linear momentum $p = m\dot{r}$, systems (6) can be written

$$(\Omega_{\mu\nu}(t, a)\dot{a}^\nu + \Gamma_\mu(t, a)) = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_l \end{pmatrix}\begin{pmatrix} \dot{r} - p/m \\ \dot{p} - f_{SA}(t, r, p) - F_{NSA}(t, r, p) \end{pmatrix} = 0 \quad (7)$$

$$a = (r, p) \in T^*M, \quad r, p \in R^n; \qquad \mu = 1, 2, \ldots, 2n$$

[14] Lagrange and Hamilton appeared to be fully aware that Newtonian forces are generally *nonderivable* from a potential. In fact, they conceived their equations with external terms representing precisely all the forces that cannot be cast into a Lagrangian or a Hamiltonian form. Oddly, it has only been since the beginning of this century that the Lagrange and Hamilton equations have been "truncated" with the removal of the external terms, acquiring the more simplistic form of contemporary use. The researcher, however, should be aware that the truncated equations, unless properly treated and presented, may literally imply the belief in perpetual motion in our environment. Intriguingly, the equations originally conceived by Hamilton *do not* have a Lie algebra structure (see below). In fact, a Lie algebra structure emerges only after the removal of the external terms.

[15] The following theorem of indirect universality of Hamilton's equations for local Newtonian systems has been proved in Vol. II of Ref. 18:

Theorem. All local, analytic, and regular Newtonian systems admit an indirect Hamiltonian representation in a star-shaped region of their variables.

The "indirect Hamiltonian representation" refers here to the representation (in terms of the "truncated" Hamilton equations) of equivalent equations of motion obtained via the use of smoothness and regularity-preserving transformations of the local variables. In essence, when representations (B2) of Appendix B do not exist (in which case the systems are called essentially nonself-adjoint) a conventional Hamiltonian representation can still be obtained via the use of the transformation theory. In this case the coordinates of the representation are (necessarily) dependent in a generally nonlinear way on the old coordinates and the momenta. As such, the coordinates are not realizable with experiments, and the representation acquires a purely mathematical meaning. *This use of Hamilton's equations will be excluded throughout this paper.* Our attitude is that, to avoid insidious traps in the physical interpretation, all abstract mathematical algorithms (H, r, p, etc.) should possess a direct physical interpretation as well as admit the coordinates actually used in experiments (which is not necessarily the case for the theorem reviewed here).

where we have included the multiplication of a regular matrix of factor terms. The conditions of variational self-adjointness for systems (7) are given by

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0 \qquad (8a)$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial a^\tau} + \frac{\partial \Omega_{\nu\tau}}{\partial a^\mu} + \frac{\partial \Omega_{\tau\mu}}{\partial a^\nu} = 0 \qquad (8b)$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial t} = \frac{\partial \Gamma_\mu}{\partial a^\nu} - \frac{\partial \Gamma_\nu}{\partial a^\mu} \qquad (8c)$$

Whenever they are satisfied in a (star-shaped) region of the local variables, system (7) is derivable from the most general possible (regular) variational principle for first-order systems, the Pfaffian principle

$$\delta^1 \int_{t_1}^{t_2} dt\, [R_\mu(t, a)\dot{a}^\mu - B(t, a)] = 0 \qquad (9)$$

The underlying analytic equations are *Birkhoff's equations*

$$\left[\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right)\right]_{SA} = 0$$
$$\det\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right) \mapsto 0 \qquad (10)$$

which admit Hamilton's equations as a particular case, i.e.,

$$\left[\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right)\right]_{SA}\Bigg|_{R=R^0} = \left(\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right)_{SA}$$
$$= \begin{pmatrix} -\dot{p} - \partial H/\partial r \\ \dot{r} - \partial H/\partial p \end{pmatrix}_{SA} = 0 \qquad (11)$$
$$B = H; \qquad R^0 = (p, 0); \qquad (\omega_{\mu\nu}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

It is possible to prove that, under sufficient smoothness conditions, there always exists a regular matrix (h) of integrating factors capable of rendering system (7) self-adjoint. This establishes the following property,[18,20] the *direct universality of Birkhoff's equations for local Newtonian systems:*

Theorem 1. All local (real) analytic, regular, Newtonian systems always admit, in a star-shaped neighborhood of a regular point of their variables, a representation in terms of Birkhoff's equations.

The Birkhoffian functions can be computed from (self-adjoint) equations of motion according to

$$R_\mu(t, a) = a^\nu \int_0^1 d\tau\, \Omega_{\mu\nu}(t, \tau a), \qquad \Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}$$
$$B(t, a) = -a^\mu \int_0^1 d\tau \left(\Gamma_\mu + \frac{\partial R_\mu}{\partial t}\right)(t, \tau a) \tag{12}$$

Algebraic Profile. A most important property of the *autonomous* Birkhoff equations is that the brackets of the time evolution law

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\dot{a}^\mu = [A, B]^* = \frac{\partial A(a)}{\partial a^\mu}\Omega^{\mu\nu}(a)\frac{\partial B(a)}{\partial a^\nu}; \quad \Omega^{\mu\nu} = (\| \Omega_{\mu\nu} \|^{-1}) \tag{13}$$

satisfy the Lie algebra laws. In particular, subsets (8a) and (8b) of the conditions of self-adjointness are equivalent to the necessary and sufficient conditions for brackets (13) to be Lie, i.e.,

$$\Omega^{\mu\nu} + \Omega^{\nu\mu} = 0$$
$$\Omega^{\mu\rho}\frac{\partial \Omega^{\nu\tau}}{\partial a^\rho} + \Omega^{\nu\rho}\frac{\partial \Omega^{\tau\mu}}{\partial a^\rho} + \Omega^{\tau\rho}\frac{\partial \Omega^{\mu\nu}}{\partial a^\rho} = 0 \tag{14}$$

The conventional Poisson brackets are clearly obtained as a particular case for $R = R^0$. Birkhoff's equations can therefore be interpreted as a Lie-algebra-preserving generalization of Hamilton's equations. The transition from the Hamilton to the Birkhoff equations is the analytic counterpart of the algebraic notion of *Lie isotopy* of Refs. 18 and 20.

The preservation of the Lie algebra character implies the existence of the integrated form of the (autonomous) Birkhoff equations, i.e.,

$$G^*(t):\quad a' = \left[\exp\left(t\Omega^{\rho\sigma}\frac{\partial B}{\partial a^\sigma}\frac{\partial}{\partial a^\rho}\right)\right] a \tag{15}$$

which is clearly a group-preserving generalization of the conventional canonical form

$$G(t):\quad a' = \left[\exp\left(t\omega^{\rho\sigma}\frac{\partial H}{\partial a^\sigma}\frac{\partial}{\partial a^\rho}\right)\right] a \tag{16}$$

The Lie-isotopic generalization of Lie's theory proposed in Ref. 20 (see Ref. 18 for a more detailed treatment and Ref. 14 for a recent review) consists of a Lie-isotopic generalization of Lie's first, second, and third theorems, complemented by a Lie-isotopic generalization of the Poincaré–

Birkhoff–Witt theorem.[16] According to this view, the Hamiltonian form (16) is a realization of the conventional formulation of Lie's theory, while the Birkhoffian form (15) is a realization of the Lie-isotopic generalization. Needless to say, these results merely establish the *existence* of a consistent, Lie-isotopic generalization of Lie's theory. Its actual detailed construction in the needed diversified form will call for contributions by a number of researchers.

Geometric Profile. Another important property of Birkhoff's equations is that subsets (8a) and (8b) of the conditions of self-adjointness coincide with the integrability conditions for an exterior two-form Ω_2 on the cotangent bundle T^*M in the local chart $a = (r, p)$ to be an exact symplectic form

$$\Omega_2 = \tfrac{1}{2}\Omega_{\mu\nu}\, da^\mu \wedge da^\nu = dR_1 = d(R_\mu\, da^\mu); \quad d\Omega_2 = 0; \quad \det(\Omega_{\mu\nu}) \neq 0 \tag{17}$$

More generally, the entire set of conditions of self-adjointness (8) coincide with the integrability conditions for a two-form $\hat{\Omega}_2$ on $\mathbf{R} \times T^*M$ to be an exact contact form, i.e.,

$$\hat{\Omega}_2 = \tfrac{1}{2}\hat{\Omega}_{ij}\,d\hat{a}^i \wedge d\hat{a}^j = d\hat{R}_1 = d(\hat{R}_i d\hat{a}^i);$$
$$d\hat{\Omega}_2 = 0; \quad d\Omega_2 = 0; \quad \hat{\Omega}_2|_{T^*M} = \Omega_2$$
$$\hat{a} = (t, a); \quad i = 0, 1, 2, \ldots, 2n; \quad \hat{\Omega}_{0\mu} = \frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}; \quad \hat{\Omega}_{\mu\nu} = \Omega_{\mu\nu} \tag{18}$$

As expected, Birkhoff's equations preserve the geometrical character of Hamilton's equations. We simply have the transition from the exact, fundamental (canonical) two-form

$$\omega_2 = \tfrac{1}{2}\omega_{\mu\nu}\, da^\mu \wedge da^\nu = dR_1^0 = d(R_\mu^0\, da^\mu) = dp_k \wedge dr^k \tag{19}$$

to the most general possible, but still exact,[17] symplectic structure.

The Birkhoffian generalization of Hamiltonian mechanics therefore can be equipped with a considerable methodological basis, consisting not only of a Lie-isotopic generalization of Lie's theory, but also of the symplectic and contact geometries.[22] Under these conditions, it is an easy prediction

[16] It should be noted here that the name "Birkhoff" in "Birkhoff's equations" and the "Poincaré–Birkhoff–Witt theorem" refer to father and son, respectively, the latter being my colleague at the Department of Mathematics, Harvard University, Prof. Garrett Birkhoff.

[17] The exactness of the forms is needed for the derivability from a variational principle as well as to preserve the geometric character of Hamilton's equations.

that the various parts of Hamiltonian mechanics will be, in due time, generalized into a Birkhoffian form. For initial studies on the Birkhoffian generalization of the canonical transformation theory, symmetries, first integrals, and related topics, we refer the reader to Ref. 18. For initial studies on a Birkhoffian generalization of the Hamilton–Jacobi theory, we refer the reader to Sarlet and Cantrijn.[23] The potential relevance of these studies for the quantum mechanical treatment of systems (6) will be indicated in the next section.

Despite these intriguing and promising properties, the Birkhoffian generalization of Hamiltonian mechanics does not appear to be the final form of mechanics, due to a number of insufficiencies. The most relevant ones are the following.

1. The brackets of the time evolution law for the *nonautonomous* Birkhoff equations

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\dot{a}^\mu = \frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}\frac{\partial B}{\partial a^\nu} + \frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}\frac{\partial R_\nu}{\partial_t} \stackrel{\text{def}}{=} A \times B \tag{20}$$

do not characterize an algebra as commonly understood (because they violate the joint, right, and left distributive laws and the scalar law). The loss of a consistent algebraic structure is clearly a drawback for the quantitative generalization of a number of conventional notions of Hamiltonian mechanics, particularly those based on Lie algebras (e.g., angular momentum). This deficiency is compounded by the fact that, even for autonomous systems (6), the computable Birkhoffian representations often depend explicitly on time.[18] In this case we cannot treat the systems considered via any algebraic structure (whether Lie or not).

2. Even though Theorem 1 ensures the existence of a Birkhoffian representation, its existence is in practice so complex (even for simple systems with low dimensionality) as to discourage even one most devoted to Lie's theory and symplectic geometry [the systems to be solved, Eq. (8) in the unknown elements $h_\mu{}^\nu$ for fixed forces, is a quasilinear, often hyperbolic system of partial differential equations].

3. The Newtonian interactions do not necessarily occur at a point, as implied by systems (6) and Eqs. (10). In actuality, physical reality demands the identification of methods which are readily and effectively applicable to arbitrary *nonlocal* systems.

Due to these (and other[18]) problematic aspects, we have sought and identified a covering of Birkhoffian (and, thus, of Hamiltonian) mechanics.

[18] For instance, the function B of Eqs. (10) cannot be, in general, the total energy. It is a mere mathematical quantity assisting in the treatment of the system. In order to differentiate the function B of Eqs. (10) from the function H of Eqs. (11), we called the former the *Birkhoffian*.[18,20] The lack of a direct physical meaning of the Birkhoffian is clearly insidious on physical grounds, particularly for quantum mechanical considerations, as we shall see better in Section 4.

3.2. The Main Idea of the Lie-Admissible Generalization of Hamiltonian Mechanics[17,20]

The most general known unconstrained Newtonian systems in Euclidean space can be written[19]

$$\{[m_k\ddot{\mathbf{r}}_k - \mathbf{f}_k(t, \mathbf{r}, \dot{\mathbf{r}})]_{SA} - \mathbf{F}_k(t, \mathbf{r}, \dot{\mathbf{r}}) - \iiint_V dv'\, \mathbf{G}(t, \mathbf{r}, \mathbf{r}', \dot{\mathbf{r}}, \dot{\mathbf{r}}', \ldots)\}_{NSA} = 0 \tag{21}$$

with a self-evident separation of the local and nonlocal forces derivable and nonderivable from a potential. Typical examples are satellites in the Earth's atmosphere, spinning tops with drag torques, etc.

The equations most effective for the representation of systems (21) are those originally conceived by Hamilton, i.e.,

$$\dot{a}^\mu - \omega^{\mu\nu}\,\partial H/\partial a^\nu - F^\mu = 0$$
$$a = (r, p), \quad H = T + V, \quad F = (0, F_{NSA}), \quad (\omega^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu = 1, 2, \ldots, 2n \tag{22}$$

rather than their truncated version (11) used in the contemporary physical and mathematical literature. In fact, Eqs. (22) possess the following properties:

(a) They are directly universal for all Newtonian systems, irrespective of their smoothness, regularity, locality, and other properties.

(b) The representation of systems (21) via Eqs. (22) is truly simple. The Hamiltonian represents the kinetic energy and the potential energy of all self-adjoint forces. All nonpotential forces, whether local or not, are represented via the external terms.

(c) Unlike the Birkhoffian case (see footnote 18), all mathematical algorithms of Eqs. (22) possess a clear and direct physical meaning. In particular, the Hamiltonian H of Eqs. (22) can always represent the total energy. Also, the variables t, r, and p can represent time, coordinates, and linear momenta in the actual experimental detection of the system.

[19] The notion of nonself-adjointness is considered here to be inclusive of that of self-adjointness. Thus, nonlocal, non-self-adjoint forces include nonlocal, self-adjoint forces.

In Ref. 20 it is noted that the brackets of the time evolution law of Eqs. (22)

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\dot{a}^\mu = \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} + \frac{\partial A}{\partial a^\mu}F^\mu \overset{\text{def}}{=} A \times H \qquad (23)$$

do not constitute an algebra as commonly understood.[20] We therefore suggested a simple modification of the way in which the external forces are written, in order to ensure the existence of a consistent algebraic structure. The reformulation of Eqs. (22) proposed in Ref. 20 reads

$$\left(\dot{a}^\mu - S^{\mu\nu}(t, a)\frac{\partial H(t, a)}{\partial a^\nu}\right) = \begin{pmatrix} \dot{r} - \partial H/\partial p \\ \dot{p} + \partial H/\partial r + s\,\partial H/\partial p \end{pmatrix} = 0,$$

$$r_k = s_{ki}\frac{\partial H}{\partial p_i}, \qquad s = s^T$$

$$S^{\mu\nu} = \omega^{\mu\nu} + T^{\mu\nu}; \qquad T^{\mu\nu} = T^{\nu\mu}; \qquad (T^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & -s \end{pmatrix} \qquad (24)$$

with a solution for all systems (21)

$$s = \operatorname{diag}\frac{F + \iiint d\nu\, G}{p/m} \qquad (25)$$

and equivalent covariant form

$$S_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} = 0; \qquad S_{\mu\nu} = \omega_{\mu\nu} + T_{\mu\nu} = (\| S^{\mu\nu} \|^{-1})_{\mu\nu}; \qquad (T_{\mu\nu}) = \begin{pmatrix} -s & 0 \\ 0 & 0 \end{pmatrix} \qquad (26)$$

It is easy to see that the brackets of the time evolution law of Eqs. (24),

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\dot{a}^\mu = \frac{\partial A}{\partial a^\mu}S^{\mu\nu}\frac{\partial H}{\partial a^\nu} \overset{\text{def}}{=} (A, H) \qquad (27)$$

satisfy the right and left distributive laws as well as the scalar law. Thus, they characterize a consistent algebra. Notice that, unlike the Birkhoffian case, this consistent algebraic structure exists for all systems (21).

Next, we proceeded to the identification of the algebra characterized by brackets (A, H), which are Lie-admissible because the attached algebra with brackets

$$(A, H) - (H, A) = 2[A, H]_{\text{Poisson}} \qquad (28)$$

[20] This is a situation similar to that of Eqs. (20).

is Lie. In this algebraic way, the truncated Hamilton equations are the attached equations of the true Hamilton equations. As a result, the truncated equations lose their fundamental role in mechanics in favor of the equations along the original unrestricted vision by (Lagrange and) Hamilton.

Since Lie-admissible algebras are known to be algebraic coverings of Lie algebras, the Lie-admissible character of Eqs. (24) permitted the identification of a new generalization of Hamiltonian mechanics. The most salient aspects are the following.

Analytic Profile. Equations (26) are derivable from the following variational principle (under sufficient smoothness and other conditions ignored here):

$$\delta^* \int_{t_1}^{t_1} dt\,[R_\mu(t, a)\dot{a}^\mu - B(t, a)]$$

$$= \int_{t_1}^{t_2} dt\left[\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right)\right]_{\text{SA}}\delta^* a^\mu$$

$$= \int_{t_1}^{t_2} dt\left(S_{\rho\sigma}(t, a)\dot{a}_\sigma - \frac{\partial H}{\partial a^\rho}\right)_{\text{NSA}}\delta a^\rho = 0; \qquad \delta^* a = g(t, a)\,\delta a \qquad (29)$$

tentatively called a *genotopically mapped Hamilton principle* (in the sense that the transition $\delta \to \delta^*$ induces a nonself-adjoint structure). The matrix (g) of Eqs. (29) is the inverse of the matrix (h) of factor functions which are needed to turn nonself-adjoint equations (26) into an equivalent self-adjoint form. The universality of principle (29) follows from the universality of the existence of this equivalence transformation.[18]

Algebraic Profile. The integrability conditions for a tensor $S^{\mu\nu}$ to characterize Lie-admissible brackets (A, H) are given by

$$(S^{\mu\rho} - S^{\rho\mu})\frac{\partial}{\partial a^\rho}(S^{\nu\tau} - S^{\tau\nu}) + (S^{\nu\rho} - S^{\rho\nu})\frac{\partial}{\partial a^\rho}(S^{\tau\mu} - S^{\mu\tau})$$

$$+ (S^{\tau\rho} - S^{\rho\tau})\frac{\partial}{\partial a^\rho}(S^{\mu\nu} - S^{\nu\mu}) = 0 \qquad (30)$$

with general solution

$$S^{\mu\nu} = \Omega^{\mu\nu} + T^{\mu\nu}; \qquad \Omega^{\mu\nu} = \left(\left\| \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \right\|^{-1}\right)^{\mu\nu}; \qquad T^{\mu\nu} = T^{\nu\mu} \qquad (31)$$